Math Formulas: Taylor and Maclaurin Series

Definition of Taylor series:

1. $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + R_n$$

2. $$R_n = \frac{f^{(n)}(\xi)(x-a)^n}{n!} \text{ where } a \le \xi \le x, \quad (\text{ Lagrangue's form })$$

3. $$R_n = \frac{f^{(n)}(\xi)(x-\xi)^{n-1}(x-a)}{(n-1)!} \text{ where } a \le \xi \le x, \quad (\text{ Cauch's form })$$

This result holds if $f(x)$ has continuous derivatives of order n at last. If $\lim_{n\to+\infty} R_n = 0$, the infinite series obtained is called **Taylor series** for $f(x)$ about $x = a$. If $a = 0$ the series is often called a **Maclaurin series.**

Binomial series

4. $$\begin{aligned}(a+x)^n &= a^n + na^{n-1} + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots \\ &= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \cdots\end{aligned}$$

Special cases of binomial series

5. $$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \quad -1 < x < 1$$

6. $$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots \quad -1 < x < 1$$

7. $$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots \quad -1 < x < 1$$

8. $$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots \quad -1 < x \le 1$$

9. $$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 + \cdots \quad -1 < x \le 1$$

Series for exponential and logarithmic functions

10. $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

11. $$e^x = 1 + x\,\ln a + \frac{(x\,\ln a)^2}{2!} + \frac{(x\,\ln a)^3}{3!} + \cdots$$

12. $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x \le 1$$

13. $$\ln(1+x) = \left(\frac{x-1}{x}\right) + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \quad x \ge \frac{1}{2}$$

Series for trigonometric functions

14. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

15. $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

16. $$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots + \frac{2^{2n}\left(2^{2n}-1\right)B_n x^{2n-1}}{(2n)!} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

17. $$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \cdots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!} \quad 0 < x < \pi$$

18. $$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{E_n x^{2n}}{(2n)!} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

19. $$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \cdots + \frac{2\left(2^{2n}-1\right)E_n x^{2n}}{(2n)!} \quad 0 < x < \pi$$

Series for inverse trigonometric functions

20. $$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \quad -1 < x < 1$$

21. $$\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots\right) \quad -1 < x < 1$$

22. $$\arctan x = \begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & -1 < x < 1 \\ \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & x \geq 1 \\ -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & x < 1 \end{cases}$$

23. $$\operatorname{arccot} x = \frac{\pi}{2} - \arctan x = \begin{cases} \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) & -1 < x < 1 \\ \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots & x \geq 1 \\ \pi + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots & x < 1 \end{cases}$$

Series for hyperbolic functions

24. $$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

25. $$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

26. $$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \frac{(-1)^{n-1}2^{2n}\left(2^{2n}-1\right)B_n x^{2n-1}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}$$

27. $$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \cdots \frac{(-1)^{n-1}2^{2n}B_n x^{2n-1}}{(2n)!} + \cdots \quad 0 < |x| < \pi$$