Analytic Geometry Formulas

1. Lines in two dimensions

Line forms

Slope - intercept form:
\[ y = mx + b \]

Two point form:
\[ y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \]

Point slope form:
\[ y - y_1 = m(x - x_1) \]

Intercept form
\[ \frac{x}{a} + \frac{y}{b} = 1 \quad (a, b \neq 0) \]

Normal form:
\[ x \cdot \cos \sigma + y \sin \sigma = p \]

Parametric form:
\[ x = x_1 + t \cos \alpha \]
\[ y = y_1 + t \sin \alpha \]

Point direction form:
\[ \frac{x - x_1}{A} = \frac{y - y_1}{B} \]

where \((A, B)\) is the direction of the line and \(P_1(x_1, y_1)\) lies on the line.

General form:
\[ Ax + By + C = 0 \quad A \neq 0 \text{ or } B \neq 0 \]

Distance

The distance from \(Ax + By + C = 0\) to \(P_1(x_1, y_1)\) is
\[ d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \]

Concurrent lines

Three lines
\[ A_1x + B_1y + C_1 = 0 \]
\[ A_2x + B_2y + C_2 = 0 \]
\[ A_3x + B_3y + C_3 = 0 \]
are concurrent if and only if:
\[ \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0 \]

Line segment

A line segment \(P_1P_2\) can be represented in parametric form by
\[ x = x_1 + (x_2 - x_1)t \]
\[ y = y_1 + (y_2 - y_1)t \]
\[ 0 \leq t \leq 1 \]

Two line segments \(P_1P_2\) and \(P_3P_4\) intersect if any only if the numbers
\[ s = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \\ x_4 - x_1 & y_4 - y_1 \end{vmatrix} \]
\[ t = \begin{vmatrix} x_3 - x_1 & y_3 - y_1 \\ x_4 - x_1 & y_4 - y_1 \end{vmatrix} \]
satisfy \(0 \leq s \leq 1\) and \(0 \leq t \leq 1\)

2. Triangles in two dimensions

Area

The area of the triangle formed by the three lines:
\[ A_1x + B_1y + C_1 = 0 \]
\[ A_2x + B_2y + C_2 = 0 \]
\[ A_3x + B_3y + C_3 = 0 \]
is given by
\[ K = \frac{1}{2} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \]

The area of a triangle whose vertices are \(P_1(x_1, y_1)\), \(P_2(x_2, y_2)\) and \(P_3(x_3, y_3)\):
\[ K = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \]
\[ K = \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \]
Centroid
The centroid of a triangle whose vertices are \( P_1(x_1, y_1) \), \( P_2(x_2, y_2) \) and \( P_3(x_3, y_3) \):
\[
(x, y) = \left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)
\]

Incenter
The incenter of a triangle whose vertices are \( P_1(x_1, y_1) \), \( P_2(x_2, y_2) \) and \( P_3(x_3, y_3) \):
\[
(x, y) = \left( \frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right)
\]
where \( a \) is the length of \( P_1P_3 \), \( b \) is the length of \( P_1P_2 \), and \( c \) is the length of \( P_2P_3 \).

Circumcenter
The circumcenter of a triangle whose vertices are \( P_1(x_1, y_1) \), \( P_2(x_2, y_2) \) and \( P_3(x_3, y_3) \):
\[
(x, y) = \left( \frac{x_1^2 + y_1^2}{x_1}, \frac{x_2^2 + y_2^2}{y_2}, \frac{x_3^2 + y_3^2}{x_3} \right)
\]

Orthocenter
The orthocenter of a triangle whose vertices are \( P_1(x_1, y_1) \), \( P_2(x_2, y_2) \) and \( P_3(x_3, y_3) \):
\[
(x, y) = \left( \frac{y_1 x_2 x_3 + y_2 x_3 x_1 + y_3 x_1 x_2}{x_1 y_2 y_3}, \frac{x_1 x_2 y_3 + x_2 x_3 y_1 + x_3 x_1 y_2}{y_1 x_2 y_3} \right)
\]

3. Circle

Equation of a circle
In an x-y coordinate system, the circle with centre \( (a, b) \) and radius \( r \) is the set of all points \( (x, y) \) such that:
\[
(x - a)^2 + (y - b)^2 = r^2
\]
Circle is centred at the origin
\[
x^2 + y^2 = r^2
\]
Parametric equations
\[
x = a + r \cos t
\]
\[
y = b + r \sin t
\]
where \( t \) is a parametric variable.
In polar coordinates the equation of a circle is:
\[
r^2 - 2rr_0 \cos(\theta - \phi) + r_0^2 = a^2
\]
Area
\[
A = r^2 \pi
\]
Circumference
\[
c = \pi \cdot d = 2\pi \cdot r
\]

Theorems:
(Chord theorem)
The chord theorem states that if two chords, CD and EF, intersect at G, then:
\[
CD \cdot DG = EG \cdot FG
\]
(Tangent-secant theorem)
If a tangent from an external point D meets the circle at C and a secant from the external point D meets the circle at G and E respectively, then
\[
DC^2 = DG \cdot DE
\]
(Secant - secant theorem)
If two secants, DG and DE, also cut the circle at H and F respectively, then:
\[
DH \cdot DG = DF \cdot DE
\]
(Tangent chord property)
The angle between a tangent and chord is equal to the subtended angle on the opposite side of the chord.
4. Conic Sections

The Parabola
The set of all points in the plane whose distances from a fixed point, called the focus, and a fixed line, called the directrix, are always equal.

The standard formula of a parabola:
\[ y^2 = 2px \]

Parametric equations of the parabola:
\[ x = 2pt^2 \\
y = 2pt \]

Tangent line
Tangent line in a point \( D(x_0, y_0) \) of a parabola \( y^2 = 2px \)
\[ y_0 = p(x + x_0) \]
Tangent line with a given slope (m)
\[ y = mx + \frac{p}{2m} \]

Tangent lines from a given point
Take a fixed point \( P(x_0, y_0) \). The equations of the tangent lines are
\[ y - y_0 = m_1(x - x_0) \text{ and} \]
\[ y - y_0 = m_2(x - x_0) \]
where
\[ m_1 = \frac{y_0 + \sqrt{y_0^2 - 2px_0}}{2x_0} \text{ and} \]
\[ m_1 = \frac{y_0 - \sqrt{y_0^2 - 2px_0}}{2x_0} \]

The Ellipse
The set of all points in the plane, the sum of whose distances from two fixed points, called the foci, is a constant.

The standard formula of an ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

Parametric equations of the ellipse
\[ x = a \cos t \]
\[ y = b \sin t \]
Tangent line in a point \( D(x_0, y_0) \) of a ellipse:
\[ \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1 \]

Eccentricity:
\[ e = \frac{\sqrt{a^2 - b^2}}{a} \]

Foci:
\[ \text{if } a > b \Rightarrow F_1(-\sqrt{a^2 - b^2},0) \quad F_2(\sqrt{a^2 - b^2},0) \]
\[ \text{if } a < b \Rightarrow F_1(0,-\sqrt{b^2 - a^2}) \quad F_2(0,\sqrt{b^2 - a^2}) \]

Area:
\[ K = \pi \cdot a \cdot b \]

The Hyperbola
The set of all points in the plane, the difference of whose distances from two fixed points, called the foci, remains constant.

The standard formula of a hyperbola:
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

Parametric equations of the Hyperbola
\[ x = \frac{a}{\sin t} \]
\[ y = \frac{b \sin t}{\cos t} \]
Tangent line in a point \( D(x_0, y_0) \) of a hyperbola:
\[ \frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1 \]

Foci:
\[ \text{if } a > b \Rightarrow F_1(-\sqrt{a^2 + b^2},0) \quad F_2(\sqrt{a^2 - b^2},0) \]
\[ \text{if } a < b \Rightarrow F_1(0,-\sqrt{b^2 + a^2}) \quad F_2(0,\sqrt{b^2 + a^2}) \]

Asymptotes:
\[ \text{if } a > b \Rightarrow y = \frac{ax}{b} \text{ and } y = -\frac{b}{a} x \]
\[ \text{if } a < b \Rightarrow y = \frac{bx}{a} \text{ and } y = -\frac{a}{b} x \]
5. Planes in three dimensions

Plane forms

Point direction form:
\[
\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}
\]
where \( P_1(x_1,y_1,z_1) \) lies in the plane, and the direction \((a,b,c)\) is normal to the plane.

General form:
\[ Ax + By + Cz + D = 0 \]
where direction \((A,B,C)\) is normal to the plane.

Intercept form:
\[
\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1
\]
this plane passes through the points \((a,0,0)\), \((0,b,0)\), and \((0,0,c)\).

Three point form
\[
\begin{vmatrix}
  x-x_3 & y-y_3 & z-z_3 \\
  x_1-x_3 & y_1-y_3 & z_1-z_3 \\
  x_2-x_3 & y_2-y_3 & z_2-z_3 \\
\end{vmatrix} = 0
\]

Normal form:
\[
x \cos \alpha + y \cos \beta + z \cos \gamma = p
\]

Parametric form:
\[
x = x_1 + a_1s + a_2t \\
y = y_1 + b_1s + b_2t \\
z = z_1 + c_1s + c_2t
\]
where the directions \((a1,b1,c1)\) and \((a2,b2,c2)\) are parallel to the plane.

Angle between two planes:
The angle between two planes:
\[
A_1x + B_1y + C_1z + D_1 = 0 \\
A_2x + B_2y + C_2z + D_2 = 0
\]
is
\[
\arccos \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}}
\]
The planes are parallel if and only if
\[
\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}
\]
The planes are perpendicular if and only if
\[
A_1A_2 + B_1B_2 + C_1C_2 = 0
\]

Equation of a plane
The equation of a plane through \( P_1(x_1,y_1,z_1) \) and parallel to directions \((a_1,b_1,c_1)\) and \((a_2,b_2,c_2)\) has equation
\[
\begin{vmatrix}
  x - x_1 & y - y_1 & z - z_1 \\
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
\end{vmatrix} = 0
\]
The equation of a plane through \( P_1(x_1,y_1,z_1) \) and \( P_2(x_2,y_2,z_2) \), and parallel to direction \((a,b,c)\), has equation
\[
\begin{vmatrix}
  x - x_1 & y - y_1 & z - z_1 \\
  x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\
  a & b & c \\
\end{vmatrix} = 0
\]

Distance
The distance of \( P_1(x_1,y_1,z_1) \) from the plane \( Ax + By + Cz + D = 0 \) is
\[
d = \frac{|Ax_1 + By_1 + Cz_1|}{\sqrt{A^2 + B^2 + C^2}}
\]

Intersection
The intersection of two planes
\[
A_1x + B_1y + C_1z + D_1 = 0 \\
A_2x + B_2y + C_2z + D_2 = 0
\]
is the line
\[
\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c},
\]
where
\[
a = \begin{vmatrix}
  B_1 & C_1 \\
  B_2 & C_2 \\
\end{vmatrix} \\
b = \begin{vmatrix}
  C_1 & A_1 \\
  C_2 & A_2 \\
\end{vmatrix} \\
c = \begin{vmatrix}
  A_1 & B_1 \\
  A_2 & B_2 \\
\end{vmatrix}
\]
\[
x_1 = \frac{D_1C_2 - D_2C_1}{a^2 + b^2 + c^2}
\]
\[
y_1 = \frac{D_2A_1 - D_1A_2}{a^2 + b^2 + c^2}
\]
\[
z_1 = \frac{D_1B_2 - D_2B_1}{a^2 + b^2 + c^2}
\]
If \( a = b = c = 0 \), then the planes are parallel.